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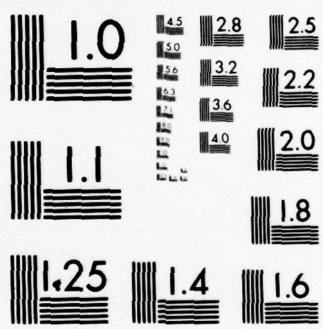

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It is known that if $T: X \to X$ is completely continuous if there exists an $n_0 > 0$ such that T^{n_0} is completely continous, then T point dissipative implies T is bounded dissipative and has a fixed point (see Billotti and LaSalle [1]). This result is used, for example, in studying retarded functional differential equations.

This result has been extended by Hale and Lopes [8]. They get the result that if T is an α-contraction and compact dissipative

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20. Abstract continued.

then T is bounded dissipative and has a fixed point. This applies, for example, to stable neutral functional differential equations and certain retarded functional differential equations of infinite delay. These results are contained in Hale [5].

The above result requires the stronger assumption of compact dissipative. The principle result of this paper will be to get similar/results under the weaker assumption of point dissipative. We will need to add additional hypotheses on the space and the operator T. We will then show how these hypotheses are naturally satisfied for stable neutral functional differential equations.

The paper will be divided into four sections. The first will contain various definitions. The second will contain an abstract theorem relating point dissipative in one space to bounded dissipative in another, and to the existence of a fixed point. The third section will apply the result to stable neutral functional differential equations. The final section gives applications to retarded functional differential equations with infinite delay.

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STABILITY AND FIXED POINTS OF POINT DISSIPATIVE SYSTEMS

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STABILITY AND FIXED POINTS OF POINT DISSIPATIVE SYSTEMS

Paul Massatt

Abstract: It is known that if $T: X \to X$ is completely continuous or if there exists an $n_0 > 0$ such that T^{n_0} is completely continuous, then T point dissipative implies T is bounded dissipative and has a fixed point (see Billotti and LaSalle [1]). This result is used, for example, in studying retarded functional differential equations.

This result has been extended by Hale and Lopes [8]. They get the result that if T is an α-contraction and compact dissipative then T is bounded dissipative and has a fixed point. This applies, for example, to stable neutral functional differential equations and certain retarded functional differential equations of infinite delay. These results are contained in Hale [5].

The ~~above~~ result requires the stronger assumption of compact dissipative. The principle result of this paper will be to get similar results under the weaker assumption of point dissipative. ~~We will~~ need exists to add additional hypotheses on the space and the operator T. ~~We will then show how~~ these hypotheses are naturally satisfied for stable neutral functional differential equations and retarded functional differential equations of infinite delay.

The paper will be divided into four sections. The first will contain various definitions. The second will contain an abstract theorem relating point dissipative in one space to bounded → to P. B.

cont dissipative in another, and to the existence of a fixed point. The third section will apply the result to stable neutral functional differential equations. The final section gives applications to retarded functional differential equations with infinite delay.

This paper is a part of my thesis at Brown University. I am especially grateful to Jack K. Hale for his help and supervision in the preparation of this paper.

STABILITY AND FIXED POINTS OF POINT DISSIPATIVE SYSTEMS

Paul Massatt

1. Definitions.

In the following definitions, X will denote a metric space, and $\mathscr{B}$ will denote the set of bounded subsets of X.

Definition 1.1: The α-measure of noncompactness is a map $\alpha: \mathscr{B} \to [0,\infty)$ defined by $\alpha(B) = \inf\{d/\text{there is a finite cover of } B \text{ with sets in } X \text{ of diameter less than } d\}$.

Definition 1.2: $T: X \to X$ is an α-contraction if there is a $k \in [0,1)$ such that for all $B \in \mathscr{B}$ we have $\alpha(TB) \leq k\alpha(B)$.

Definition 1.3: $T: X \to X$ is α-condensing if for all $B \in \mathscr{B}$ we have $\alpha(TB) \leq \alpha(B)$ with equality if and only if $\alpha(B) = 0$.

Definition 1.4: A measure of noncompactness on X is a map $\beta: \mathscr{B} \to [0,\infty)$ with the two properties

(i) $\beta(B) = 0$ if and only if $Cl(B)$ is compact and

(ii) $\beta(B) \leq \beta(C)$ if $B \subset C$.

The definitions of a β-contraction and β-condensing map are analogous to those of an α-contraction and α-condensing map.

Definition 1.5: Let $T: X \to X$ and let S be a collection of sets. A bounded set $B \subset X$ dissipates S-sets under T if for any $C \in S$ there is an integer $n_0(C)$ such that $T^n(C) \subset B$ for $n \geq n_0(C)$. If $S = \{\{x\}/x \in X\}$, we say T is point dissipative. If $S = \{J \subset X/J$ is compact$\}$, we say T is compact dissipative. If S contains a neighborhood of any point, we say T is local dissipative. If S contains a neighborhood of any compact set, we say T is local compact dissipative. If S contains all bounded subsets of X, we say T is bounded dissipative or ultimately bounded.

Remark 1.1: Local dissipative and local compact dissipative are always equivalent. If T is continuous then local dissipative and compact dissipative are also equivalent.

Definition 1.6: The orbit of $B, \gamma^+(B)$, for $B \subset X$, is defined by $\gamma^+(B) = \bigcup_{n=0}^{\infty} T^n(B)$

Definition 1.7: $H\colon \mathcal{B} \to \mathcal{B}$ is a type 2 set operator if for any $B \in \mathcal{B}$, $H(B) = \cup\{H(A)/A$ is a finite subset of $B\}$.

Definition 1.8: Let $B \in \mathcal{B}$. The orbit of B under H, $\gamma_H^+(B)$, is defined $\gamma_H^+(B) = \bigcup_{n=0}^{\infty} H^n(B)$.

Definition 1.9: $H\colon \mathcal{B} \to \mathcal{B}$ is asymptotically smooth if, for any $B \in \mathcal{B}$ with $\gamma_H^+(B)$ bounded, there is a compact set J such that $H^n(B) \to J$ (i.e. for every $\varepsilon > 0$ there exists an n_0 such that $n \geq n_0$ implies $H^n(B) \subset J + B_\varepsilon(0)$, where $B_\varepsilon(0)$ is the ball of radius ε centered at 0).

Definition 1.10: A set $K \in \mathcal{B}$ is stable under H if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that, for any

$B \subset K + B_\delta(0)$, $H^n(B) \subset K + B_\varepsilon(0)$ for $n \geq 0$.

Definition 1.11: A set $K \in \mathscr{B}$ is uniformly asymptotically stable if it is stable and there is a $\delta_0 > 0$ such that for any $\varepsilon > 0$ there is an n_0 such that $B \subset K + B_{\delta_0}(0)$ implies $H^n(B) \subset K + B_\varepsilon(0)$ for $n \geq n_0$.

Definition 1.12: A set $J \in \mathscr{B}$ is invariant under T if $TJ = J$.

2. A General Theorem.

Theorem 2.1: Let $i: X_1 \hookrightarrow X_2$ be a compact imbedding where X_i are Banach spaces with norms $||\cdot||_i$. Let T, C, and U be continuous operators mapping X_i into itself. Denote the topology of X_i by $\mathscr{T}_i$. Let $T = C + U$ with C a contraction in X and $U: (X_1, \mathscr{T}_2) \to (X_1, \mathscr{T}_1)$ mapping bounded sets to bounded sets. Let $C(0) = 0$. Let $B_R^i = \{x \in X/||x||_i < R\}$. Then the following conclusions holds:

(1) If $B \subset B_L^1$ and $R > 0$ then there exists a $K = K(L,R)$ such that, for any n^* with $0 \leq n^* \leq \infty$, then $\bigcup_{0 \leq m \leq n^*} T^m(B) \subset B_R^2$ implies $\bigcup_{0 \leq m \leq n^*} T^m(B) \subset B_K^1$.

(2) If T is point dissipative in X_2, then T is bounded dissipative in X_1.

Remark 2.1: It is not necessary to assume that $U: (X_1, \mathscr{T}_2) \to (X_1, \mathscr{T}_1)$ takes bounded sets into bounded sets. The conclusions are naturally weaker and hold only for those bounded sets whose images under U are bounded.

<u>Proof of 1</u>: Let $h: R^+ \to R^+$ be chosen so that $U(X_1 \cap B_R^2) \subset B^1_{h(R)}$. Suppose $B \subset B^1_L$ and $\bigcup_{0 \le m \le n^*} T^m(B) \subset B^2_R$. Let $\lambda \in [0,1)$ be the contraction constant for C. Let $K(L,R) = (1-\lambda)^{-1}(h(R) + L)$. If $x \in B$ we can easily show by induction $\bigcup_{0 \le m \le n^*} T^m(B) \subset B^1_K$. It is obviously true for $n = 0$. If it is assumed true for n, then

$$\begin{aligned}
||T^{n+1}x||_1 &\le \lambda||T^n x||_1 + h(R) \\
&\le \lambda(1-\lambda)^{-1}(h(R) + L) + h(R) \\
&\le \lambda(1-\lambda)^{-1}(h(R) + L) + (h(R) + L) \\
&\le (1-\lambda)^{-1}(h(R) + L) = K(L,R).
\end{aligned}$$

Hence, $\bigcup_{0 \le m \le n^*} T^m(B) \subset B^1_K$.

We shall need the following lemma before proceeding.

<u>Lemma 2.1</u>: Under the hypotheses of Theorem 2.1, there exists a number $n_1(L,R)$ and a number $Q(R) > 0$ such that if $\bigcup_{0 < m < n^*} T^m(B) \subset B^1_2 \cap B^2_L$ then $\bigcup_{n_1(L,R) \le m \le n^*} T^m(B) \subset B^1_{Q(R)}$. Here we have n_1 independent of n^* and Q independent of both n^* and L.

<u>Proof of Lemma</u>: Let $\delta_0 > 0$ and $Q(R) = (1-\lambda)^{-1}h(R) + \delta_0$.

If $B \subset X_i$ is bounded let $||B||_i = \sup\{||x||_i / x \in B\}$. Also,

$K(0,R) = (1-\lambda)^{-1}h(R)$. If $L \le K(0,R)$ we obtain the conclusion by letting $n_1 = 0$ and applying the same inductive argument used to prove part 1 of the theorem. If $L > K(0,R)$ choose n_1 large enough so that $\lambda^{n_1}(L-K(0,R)) < \delta_0$. Now we notice

$$\begin{aligned} ||TB_L^1||_1 &\le \lambda L + h(R) \\ &= \lambda(L+K(0,R)) + \lambda(1-\lambda)^{-1}h(R) + h(R) \\ &= \lambda(L-K(0,R)) + (1-\lambda)^{-1}h(R) \\ &= \lambda(L-K(0,R)) + K(0,R) \end{aligned}$$

By induction we get for $n \le n^*$

$$||T^n B_L^1||_1 \le \lambda^n(L-K(0,R)) + K(0,R).$$

Hence, for $n_1 \le n \le n^*$ we have $||T^n B_L^1||_1 \le K(0,R) + \delta_0 = Q(R)$. This completes the proof of the lemma.

Proof of Part 2 of the Theorem:

Let B_k^2 dissipate points in X_1. By part (1), it is clear that orbits of points are bounded in X_1. Since the orbit of any point is eventually dissipated by B_R^2, Lemma 2.1 implies that $B_{Q(R)}^1$ dissipates points in X_1.

We will now show $\gamma^+(B_{Q(R)}^1)$ is bounded in X_1. Let $x \in B_{Q(R)}^1$ and $\gamma^+(x) \subset B_{C(x)}^2$. We may find a constant $C(x)$ since point dissipative in X_2 implies orbits of points are bounded in X_2. Then we have $\gamma^+(x) \subset B^1_{K(Q(R),C)}$ by part 1. Let $n_0(x)$ be chosen so that if $n > n_0(x)$ then $T^n x \in B^2_R$. Let $n_1(x) = n_1(K(Q(R),C(x)),R)$. Let $n^*(x) = n_0(x) + n_1(x)$. By continuity we may choose $\delta(x)$ so that $\bigcup_{0 \le m \le n^*(x)} T^m(B^2_\delta(x) \cap B^1_{Q(R)}) \subset B^2_{C(x)}$ and

$\bigcup_{n_0(x) < m \le n^*(x)} T^m(B^2_\delta(x) \cap B^1_{Q(R)}) \subset B^2_R$ where

$B^i_\delta(x) = \{y \in X_i / ||y-x||_i < \delta\}$. Then part 1 implies

$\bigcup_{0 \le m \le n^*} T^m(B^2_\delta(x) \cap B^1_{Q(R)}) \subset B^1_{K(Q(R),C(x))}$ and Lemma 2.1 implies

$T^{n^*(x)}(B^2_\delta(x) \cap B^1_{Q(R)}) \subset B^1_{Q(R)}$.

Since $B^1_{Q(R)}$ is a compact set in X_2, the sets $B^1_{Q(R)} \cap B^2_\delta(x)$ form an open cover for which there is a finite subcover $\{B^1_{Q(R)} \cap B^2_\delta(x_i)\}^m_{i=1}$. Let $N = \max\{n^*(x_i)\}^m_{i=1}$. Then it is clear that $\gamma^+(B^1_{Q(R)}) = \bigcup_{0 \le m \le N} T^m(B^1_{Q(R)})$ since any point in $B^1_{Q(R)}$ returns to $B^1_{Q(R)}$ by the N^{th} iteration.

To show any set B^1_A is dissipated by $\gamma^+(B^1_{Q(R)})$ we use the same type of argument. We form an open cover of B^1_A by neighborhoods $\{B^2_\delta(x) \cap B^1_A\}$ such that there is an $n^*(x)$ such that $T^{n^*(x)}(B^2_\delta(x) \cap B^1_A) \subset B^1_{Q(R)}$. Then we take a finite subcover $\{B^2_\delta(x_i) \cap B^1_A\}^m_{i=1}$ and let $N = \max\{n^*(x_i)\}^m_{i=1}$. Clearly $n > N$ implies $T^n(B^1_A) \subset \gamma^+(B^1_{Q(R)})$. Since the argument is analogous I have only sketched the proof. This completes the proof.

<u>Corollary 2.1</u>: Suppose the assumptions of Theorem 2.1, T is point dissipative in X_2 and β-condensing in X_1 with β a

measure of noncompactness satisfying the property $\beta(A \cup B) = \beta(A)$ if B is a finite set. Then there exists a maximal compact invariant set in X_1 which is uniformly asymptotically stable.

<u>Proof</u>: Theorem 2.1 implies that T is bounded dissipative. Let B_R^1 dissipate bounded sets. Hence, orbits of bounded sets are bounded. Since T is β-condensing, T is asymptotically smooth (see Massatt [10]). For any bounded set B there is a compact invariant set $\omega_B \subset B_R^1$ that attracts B. Let $\omega = \bigcup_{B \in \mathcal{B}} \omega_B$. Then $\omega \subset B_R^1$ and ω is invariant. Since T is β-condensing this implies ω is precompact. Clearly ω attracts all bounded sets. ω is stable since otherwise there would exist an $\varepsilon > 0$ and sequences $\{x_k\} \subset X_1$, $\{n_k\} \to \infty$ such that $d(x_k, \omega) \to 0$ and $d(T^{n_k} x_k, \omega) > \varepsilon$ for all k. But $\{x_k\}$ is a bounded set and so ω attracts $\{x_k\}$. This is clearly a contradiction. The fact that ω is stable and attracts bounded sets clearly implies that ω is uniformly asymptotically stable. Q.E.D.

<u>Corollary 2.2</u>: Under the hypotheses of Theorem 2.1, let T be β-condensing in X_1 and point dissipative in X_2 with β-satisfying the following properties:

(i) $\beta(\overline{co}\ A) = \beta(A)$ and

(ii) $\beta(A \cup B) = \max[\beta(A), \beta(B)]$. Then T has a fixed point.

<u>Proof</u>. It is already known that if T is β-condensing and compact dissipative it has a fixed point (see Massatt [10] for general measures of noncompactness and Hale and Lopes [8], Nussbaum [12] for α-measures).

Since bounded dissipative implies compact dissipative, the corollary is proved.

3. Applications to Neutral Equations

In this section, we apply the results of Section 2 to show that a point dissipative periodic stable neutral functional differential equation has a periodic solution and that the period map is bounded dissipative in W_1^∞. We will also get the existence of a maximal compact invariant set in W_1^∞ which is uniformly asymptotically stable, with respect to the period map T.

Previously, the existence of a fixed point was only known under the assumption of compact dissipative. Under this assumption, the existence of a maximum compact invariant set in C which is uniformly asymptotically stable is also known. (See Hale [5]).

For $r > 0$, let $C = C([-r,0],\mathbb{R}^n)$ be the space of continuous functions from $[-r,0]$ to $\mathbb{R}^n$ with the supremum norm. Let $W_1^\infty = W_1^\infty([-r,0],\mathbb{R}^n)$ be the space of absolutely continuous functions with derivative essentially bounded. Let

$$||\phi||_{W_1^\infty} = \sup_{[-r,0]} |\phi(\theta)| + \operatorname*{ess\,sup}_{[-r,0]} |\phi'(\theta)|$$

If $x(\cdot): [-r,A) \to \mathbb{R}^n$, $A \geq 0$, let $x_t(\cdot): [-r,0] \to \mathbb{R}^n$ be defined by $x_t(\theta) = x(t+\theta)$ for $t \in [0,A)$, $-r \leq \theta \leq 0$. Suppose $D: C \to \mathbb{R}^n$ is linear and $D\phi = \phi(0) - L\phi$ where L is nonatomic at zero. Suppose $f: \mathbb{R}^+ \times C \to \mathbb{R}^n$ is completely continuous. A neutral functional differential equation (NFDE) is a relation

$$\frac{d}{dt} Dx_t = f(t,x_t). \tag{3.1}$$

A stable neutral functional differential equation (SNFDE) is a NFDE for which D is stable; that is, the zero solution of $Dx_t = 0$ is uniformly asymptotically stable. We often use properties of these equations without proof. The details can be found in Hale [5].

Our first result deals with continuous dependence in W_1^∞.

<u>Theorem 3.1</u>: If the solutions of (3.1) are uniquely defined by the initial data then, for $t > t_0$, the solution map $X_{D,f}(t,t_0)$: $W_1^\infty \to W_1^\infty$ given by $X_{D,f}(t,t_0)\phi = x_t$, $x_{t_0} = \phi$, is continuous in ϕ. It is also continuous with respect to f with the topology of uniform convergence on bounded sets of $\mathbb{R}^+ \times C$.

In general, there is no continuous dependence with respect to (t,t_0). This was the motivation for Melvin [11] to discuss continuous dependence in W_1^∞ in the weak-*-topology.

<u>Proof</u>: To prove the theorem we need only prove it for time $\tau > 0$ arbitrarily small, since for larger times, we may proceed by steps.

For τ small enough we first prove $X_{D,f}(\tau,0)$: $W_1^\infty \to W_1^\infty$. The proof is based on the Schauder fixed point theorem. Let $x(\cdot)$: $[-r,\tau] \to \mathbb{R}^n$ be the solution with initial condition $\phi \in W_1^\infty$. Let $z(\cdot) \in W_1^\infty[0,\tau]$. Let $D\phi = \phi(0) + L\phi$ with L nonatomic. Let ϕ^τ: $[-r,\tau] \to \mathbb{R}^n$ be defined by

$$\phi^{\tau}(\theta) = \begin{cases} \phi(\theta), & \theta \in [-r,0] \\ \phi(0), & \theta > 0 . \end{cases}$$

Let $z_t\colon [-r,0] \to \mathbb{R}^n$ be defined by

$$z_t(\theta) = \begin{cases} z(t+\theta) & \theta + t \geq 0 \\ 0 & \theta + t < 0 . \end{cases}$$

If

$$x(t) = \begin{cases} \phi(t) & t < 0 \\ z(t) + \phi(0) & t \geq 0 . \end{cases}$$

Is a solution of (3.1) then $z(\cdot)$ must be a fixed point of T where $(Tz)(t) = -Lz_t - L\phi_t^{\tau} + L\phi + \int_0^t f(s,z_s)ds$. Since f is completely continuous and L is a contraction on $z(\cdot)$ for small enough τ, there exists a $\tau > 0$, a closed, bounded, convex set $B \subset W_1^{\infty}$ such that $TB \subset B$. Since B is compact in C and T is continuous in C we have a fixed point by the Schauder fixed point theorem.

To prove continuous dependence in W_1^{∞} is a similar argument. Given an initial function $\phi_0 \in W_1^{\infty}$ and a solution $x^0(t)$ defined on $[-r,\tau]$ with $\tau > 0$, $x_0^0 = \phi_0$, we show continuous dependence with respect to ϕ_0 and f. If $x(t) = x^0(t) + z(t) + \phi(0)$ for $t \in [0,\tau]$, $x_0^0 = \phi_0$, is the solution of (3.1) with initial condition $\phi_0 + \phi$ then $z(t)$ is a fixed point of

$$(Tz)(\theta) = -Lz_t - L\phi_t^\tau + L\phi + \int_0^t [\, f(s,x_s^0+\phi_s^\tau+z_s) - f(s,x_s^0)]ds.$$

For τ small enough, L is a contraction on z and there is an $E \in \mathbb{R}^+$ such that $TB_E^1 \subset B_E^1$. Since B_E^1 is compact in C we have a fixed point. Futhermore, as $||\phi||_{W_1^\infty} \to 0$ we may let $E \to 0$. Since the fixed point must be in B_E^1, the uniqueness implies the continuous dependence on f. The proof for continuous dependence on f is analogous, and so the proof is omitted.

<u>Theorem 3.2</u>: *A SNFDE given by (3.1) with f completely continuous, periodic in t of period $\omega > 0$ and whose solution map $X_{D,f}(\omega,0)\colon C \to C$ is point dissipative, has a periodic solution of period ω.*

<u>Proof</u>: We only prove the theorem under the hypothesis that the solution map takes bounded sets into bounded sets. The proof without this hypothesis is just a technical modification using the remark after Theorem 2.1.

Let $X_1 = C[-r,0]$ and $X_2 = W_1^\infty[-r,0]$. Let $X_{D,f}(\omega) = X_{D,f}(\omega,0)$ be the period map. Let $T_{D,h}(\omega)$ be the solution map of the difference equation $Dy_t = h(t)$. According to the theory in Hale [5] there are projection maps P and $Q = I - P$, with Q finite dimensional and $P\colon C \to Y_0$ where $Y_0 = \{\phi \in C/D\phi = 0\}$. By the construction, the maps P and Q map X_1 into itself also. If we let $T_{D_0}\colon Y_0 \to Y_0$ be the solution map of $Dy_t = 0$ then, since D is stable, $||T_{D_0}^n(\omega)|| \le K\lambda^n$ for some $K > 0$, $\lambda \in [0,1)$.

We may now extend $T_{D_0}: Y_0 \to Y_0$ to $T_{D_0}: L^\infty \to L^\infty$, and still get $||T^n_{D_0}(\omega)|| \leq K\lambda^n$. This is done by using the uniqueness of the $X_{D,0}(\omega)$ map. Let $z_k \in Y_0$, $y_0 \in L$ and $\int_{-r}^{\theta} z_k \to \int_{-r}^{\theta} y_0$ in C, with $||z_k||_C \leq ||y_0||_\infty$. Then $\{X_{D,0}(\omega) \int_{-r}^{\theta} z_k\} \to X_{D0}(\omega) \int_{-r}^{\theta} y_0$ in C. But $|\frac{d}{d\theta} \int_{-r}^{\theta} z_k| \leq ||y_0||_\infty$ so for all k we get

$$||\frac{d}{d\theta} X^n_{D,0}(\omega) \int_{-r}^{\theta} z_k|| = ||T^n_{D_0}(\omega) z_k|| \leq K\lambda^n ||z_k|| \leq K\lambda^n ||y_0||_\infty .$$

Hence, $||T^n_{D_0}(\omega) y_0|| \leq K\lambda^n ||y_0||_\infty$.

Since D is stable we know that $X_{D,0}(\omega)$ is a stable operator in X_2. Also, from the theory in Hale $\lim_{n\to\infty} X^n_{D,0}(\omega)$ *exponentially* approaches a finite dimensional subspace of C generated by the constant functions. We will call this map $g(\cdot): C \to C$. g may also be considered as a map $g: X_1 \to X_1$ which is continuous, finite dimensional and satisfies $g: (X_1, \mathcal{F}_2) \to (X_1, \mathcal{F}_1)$ maps bounded sets to bounded sets.

Let $C = X_{D,0}(\omega)P(\cdot) - g(P(\cdot))$. Because of the exponential decay, there is an equivalent norm in X_1 where C is a contraction.

Now, let $U = X_{D,f}(\omega) - C$. We need to show $U: (X_1, \mathcal{F}_2) \to (X_1, \mathcal{F}_1)$ maps bounded sets to bounded sets and that $X_{D,f}(\omega): X_1 \to X_1$ is condensing. Notice $U\phi = H\phi + g(P\phi)$ where $H\phi$ is the solution at time ω to the equation $\frac{d}{dt} Dy_t = h_\phi(t)$ with initial condition $y_0 = Q\phi$ and $h_\phi(t) = f(t,x_t)$ where x_t solves (3.1) with $x_0 = \phi$. Recall that we are assuming

$X_{D,f}(\omega): X_2 \to X_2$ maps bounded sets to bounded sets. Let B be a bounded set in X_2. Since f is completely continuous, this insures $\{h_\phi(t)/\phi \in B\}$ are bounded uniformly on $[0,\omega]$. Since solutions to $Dy_t = h_\phi(t)$, $\dot{y}_0 = \frac{d}{d\theta}[Q\phi]$ are bounded we get $U: (X_1, \mathscr{F}_2) \to (X_1, \mathscr{F}_1)$ maps bounded sets to bounded sets.

Next, we will show U is completely continuous in X_1. Once we have this it is fairly easy to see $X_{D,f}(\omega): X_1 \to X_1$ is an α-contraction.

For any $L > 0$ we must show $U(B_L^1)$ is precompact. Since $\mathrm{Cl}\ B_L^1$ is compact in X_2 we also get $\mathrm{Cl}\{x_t/t \in [0,\omega],\ x_0 \in B_L^1\}$ is compact in X_2. This implies $\{h_\phi(t)/\phi \in B_L^1\}$ lie in a compact subset of $C[0,\omega]$ since $h_\phi(t) = f(t,x_t)$. By our continuous dependence theorem for W_1^∞ this implies $\mathrm{Cl}\ H(B_L^1)$ is compact in X_1. Since g also has finite dimensional range, we get $U: X_1 \to X_1$ is completely continuous.

Hence, $X_{D,f}(\omega)$ is an α-contraction in some equivalent norm in X_1. Since all the conditions of Theorem 2.1 are now satisfied we have a fixed point for $X_{D,f}(\omega)$, and hence a periodic solution of (3.1) of period ω.

<u>Remark 3.1</u>: Under the assumption that $X_{D,f}(\omega): X_2 \to X_2$ takes bounded sets to bounded sets, the above proof also shows $X_{D,f}(\omega): X_1 \to X_1$ is bounded dissipative and there is a maximal compact invariant set in X_1 which is uniformly asymptotically stable.

<u>Remark 3.2</u>: The conclusions still hold if the operator D in the NFDE is ω-periodic in t. There are only minor technical changes

in the proof for this case.

4. Applications to RFDE's with Infinite Delay.

In this section, we study the existence of periodic solutions of the equation

$$\dot{x}(t) = f(t,x_t) \qquad (*)$$

where $x_t(\mu) = x(t+\mu)$, $\mu \in (-\infty,0]$, $\mathcal{F}: \mathbb{R}^+ \times X \to \mathbb{R}^n$ is completely continuous, X is a space of functions satisfying certain axioms to be specified below, and there is an $\omega > 0$ such that $f(t,\phi) = f(t+\omega,\phi)$ for all t,ϕ.

Our objective is to show that point dissipative systems have an ω-periodic solution. There are two notions of point dissipative for these systems. The usual one is the phase space X and the other is $\mathbb{R}^n$. Let $x(t,\phi)$ be the solution of (*) with $x(t,\phi) = \phi(t)$ for $t \in (-\infty,0]$. System (*) is point dissipative in $\mathbb{R}^n$ if there is a bounded set $B_R \subset \mathbb{R}^n$ such that for any $\phi \in X$ there is a $t_0[\phi] > 0$ such that $x(t,\phi) \in B_R$ for $t > t_0(\phi)$.

We will shortly give axioms which will show what phase spaces are permissible. From the axioms one can deduce that point dissipative in X implies point dissipative in $\mathbb{R}^n$, but the converse need not hold. An example will be given later to illustrate this point. Our theorem will be proved for point dissipative in $\mathbb{R}^n$.

We will first prove results for the space $C_0^\gamma = \{\gamma(\cdot): (-\infty,0] \to \mathbb{R}^n / ||x|| = \sup_{\theta\in(-\infty,0)} |e^{\gamma\theta}x(\theta)| < \infty,\ \lim_{\theta\to-\infty} |e^{\gamma\theta}x(\theta)| = 0\}$ with $\gamma > 0$. This should help the reader understand some of the more general results which will be given later. Let

$$W_1^{\infty,\gamma'} = \{x(\cdot): (-\infty,0] \to \mathbb{R}^n / ||x|| = \sup_{\theta\in(-\infty,0]} |e^{\gamma'\theta}x(\theta)| + ||e^{\gamma'\theta}\dot{x}(\theta)||_\infty < \infty\}$$

<u>Theorem 4.1</u>: If the system (*) is point dissipative in $\mathbb{R}^n$ and $X = C_0^\gamma$ then if $\gamma' > \gamma > 0$ the period map is bounded dissipative in $W_1^{\infty,\gamma'}$ (if it maps bounded sets to bounded sets), there is a maximal compact invariant set which is uniformly asymptotically stable in $W_1^{\infty,\gamma'}$ and it has a fixed point.

<u>Proof</u>: Let $X_2 = C_0^\gamma$, $X_1 = W_1^{\infty,\gamma'}$. We will show Theorem 2.1, Corollary 2.1, and Corollary 2.2 apply. Clearly $i: X_1 \hookrightarrow X_2$ is a compact imbedding. The solution map shall be denoted $T_f(\omega)$. Clearly $T_f(\omega): X_1 \to X_1$ is continuous. We shall assume $T_f(\omega)$ maps bounded sets to bounded sets (in C_0) to avoid technical details. Let $C\phi = \phi^\omega - \phi(0)$ where $\phi(0)$ is the constant function. C is clearly a contraction. Since f is completely continuous, by arguments similar to the last theorem we get $U: (X_1, \mathcal{F}_2) \to (X_1, \mathcal{F}_1)$ maps bounded sets to bounded sets and that $U: X_1 \to X_1$ is completely continuous. This also proves that $T_f(\omega)$ is an α-contraction. All the hypotheses are now

satisfied for applying Theorem 2.1, Corollary 2.1, and Corollary 2.2. Q.E.D.

Remark 4.1: There are many other spaces for which this theorem could apply. For example, we could let $X_2 = L^2(g) \times \mathbb{R}^n =$ $\{x(\cdot): (-\infty,0] \times \mathbb{R}^n / ||g(\cdot)x(\cdot)||_2 + ||x(0)|| = ||x|| < \infty\}$ and $X_1 = W_1^2(g') \to \mathbb{R}^n = \{x(\cdot): (-\infty,0] \to \mathbb{R}^n / ||x|| = ||g'(\cdot)x(\cdot)||_2 +$ $||g'(\cdot)\dot{x}(\cdot)||_2 + ||x(0)|| < \infty\}$ with $0 < g(t) < g'(t) \le Ke^{at}$ for some $K > 0$, $a > 0$, g,g' monotonically increasing and $\lim_{t\to-\infty} \frac{g(t)}{g'(t)} = 0$.

The local axioms on the phase space will be derived from Schumacher [13]. The global axioms will come for the most past from Hale and Kato [6].

Local Axioms: Let X be a metrizable topological vector space of functions $x: (-\infty,0] \to \mathbb{R}^n$ with Hausdorff equivalence $\sim$. If $x,y \in X$ we say $x \sim y$ if there does not exist two open disjoint sets A and B with $x \in A$ and $y \in B$. Let X also satisfy the following axioms:

(1) For all $y \in X$, $\sigma \in \mathbb{R}^+$, the static continuation y^σ, defined by

$$y^\sigma(\mu) = \begin{cases} y(\sigma+\mu) & -\infty < \mu < -\sigma \\ y(0) & -\sigma \le \mu \le 0 \end{cases}$$

belong to X.

(2) $y \sim z$ implies $y^\sigma \sim z^\sigma$

(3) $\delta: X \to \mathbb{R}^n$ defined by $\delta x = x(0)$ is continuous

(4) The map $\sigma \in \mathbb{R}^+ \to y^\sigma \in X$ is continuous for all $y \in X$.

(5) For all $\tau > 0$, the set $C_\tau = \{x(t): x$ is continuous with support in $[-\tau,0]\}$ belongs to X.

(6) If C_τ has the sup norm then $i: C_\tau \to X$, the inclusion map, is continuous for all $\tau > 0$.

Axiom 3 is stronger than Schumacher's stated assumption but the author feels Schumacher needs this assumption for his results.

With these assumptions and appropriate hypotheses on f one can prove existence, uniqueness, continuous dependence, and continuation theorems. (See Hale and Kato [6] or Schumacher [13].) We shall always assume f satisfies enough hypotheses to insure these properties.

Global Axioms:

(1) All continuous bounded functions are in X. The space of continuous bounded functions with the sup norm will be denoted C_B.

(2) Let $B^\sigma = \{x^\sigma/x \in B\}$. If $B \subset X$ is bounded and $x \in B$ implies $\delta(x) = 0$ then $B^\sigma \to 0$ as $\sigma \to \infty$.

(3) The map $i: C_B \to X$ is continuous.

Statement 2 may be stronger than one desires, and spaces have been suggested where (2) does not hold. A weaker version of (2) which suffices for our purposes is as follows.

(2') If $B \subset C_B$ is bounded in sup norm and $x \in B$ implies $\delta(x) = 0$ then $B^\sigma \to 0$ in X.

Examples.

(1) Let $g(\cdot): (-\infty,0] \to \mathbb{R}^+$ be continuous, monotone increasing, and there exists $K > 0$, $a > 0$ such that $g(t) \leq Ke^{at}$. Let $X = \{x(\cdot)/x(\cdot): (-\infty,0] \to \mathbb{R}^n$ is continuous and

$$||x|| = \lim_{t\in(-\infty,0)} |g(t)x(t)| < \infty\}.$$

For Axiom 2' we may replace the condition $g(t) \leq Ke^{at}$ by $\lim_{t\to-\infty} g(t) = 0$.

(2) Let g be monotone increasing with $\int_{-\infty}^{0} g < \infty$. Let $r \geq 0$ and for any locally measurable $\phi: (-\infty,0] \to \mathbb{R}^n$ whose restriction to $[-r,0]$ is continuous let

$$||\phi||_p = \{ \sup_{-r\leq\theta\leq 0} |\phi(\theta)|^p + \int_{-\infty}^{0} g(\theta)|\phi(\theta)|^p d\theta\}.$$

Let X denote the space of such functions with $||\phi||_p < \infty$. This satisfies Axiom 2'. For Axiom 2 we may add the condition that there exists a $K > 0$, $a > 0$ such that $g(t) \leq Ke^{at}$.

(3) Let X be the space of continuous functions $x(\cdot): (-\infty,0] \to \mathbb{R}^n$ with the compact open topology.

Theorem 4.2: If the system (*) is point dissipative in $\mathbb{R}^n$ and X is a Banach space satisfying global Axioms 1,2, and 3 then there is an ω-periodic solution of (*). Also, for some $\gamma' > 0$ the space $W_1^{\infty,\gamma'}$ may be compactly imbedded into X and is bounded dissipative (if the solution map sends bounded sets to bounded sets) and there exists a compact invariant set which is uniformly asymptotically stable.

Proof: To prove this result we only need show there is a $\gamma > 0$ and an imbedding $i: C_0^\gamma \to X$ which is continuous. Then we may apply Theorem 4.1.

Let $B = \{x \in C_B / x(0) = 0, |x| < 1\}$. Let $||B||_X = \lambda > 0$ where $||B||_X = \sup\{||x||_X / x \in B\}$. Let $\sigma > 0$ be chosen so that $||B^\sigma|| = \rho_0\lambda$ for some $\rho_0 < 1$. Choose ρ_1 so that $\rho_0 < \rho_1 < 1$ and let $\gamma > 0$ be chosen so that $e^{-a\gamma} = \rho_1$. I claim $i: C_0^\gamma \to X$ is a continuous imbedding. Since C_B is dense in C_0^γ and any sequence of elements in $C_0^\gamma \cap C_B$ which is Cauchy in C_0^γ is also Cauchy in X. We get that $i: C_0^\gamma \hookrightarrow X$ is a continuous imbedding. Q.E.D.

We now state the main result of this section.

Theorem 4.3: If system (*) is point dissipative in $\mathbb{R}^n$ then there is an ω-periodic solution of (*). This theorem removes the restriction that X be a Banach space and replaces the condition that X satisfy global Axiom 2 with the weaker global Axiom 2'. The proof will use the following two results.

Theorem 4.4 (Horn's Theorem [9]): Let $S_0 \subset S_1 \subset S_2$ be convex subsets of a Banach space X with S_0, S_2 compact and S_1 open in S_2. Let $T: S_2 \to X$ be a continuous mapping such that, for some integer $m > 0$, $T^j S_1 \subset S_2$ for $0 \leq j \leq m - 1$ and $T^j S_1 \subset S_0$

for $j \geq m$. Then T has a fixed point.

The proof of the next theorem is identical to the proof of a Theorem 4.1 in Massatt [10]. The proof will be given here, though.

Theorem 4.5: Let X_0 and X_1 be two Banach spaces with a continuous imbedding $i: X_0 \hookrightarrow X_1$. Let $\mathscr{B}_i$ be the collection of bounded sets in X_i. Let $H: \mathscr{B}_0 \to \mathscr{B}_0$ be a type 2 set operator. Let $\beta: \mathscr{B}_0 \to [0,\infty)$ be a map satisfying the following properties.

(1) If $B \in \mathscr{B}_0$ then $\beta(B) = 0$ if and only if $\mathrm{Cl}_{X_1}(B)$ is compact in X_1.

(2) Let $A, B \in \mathscr{B}_0$ and let B be a finite set, then $\beta(A \cup B) = \beta(A)$. Under these conditions, if H is β-condensing then H restricted to $\mathscr{B}_0$ is asymptotically smooth in X_1.

Proof of Theorem 4.4: Let $\gamma^+(B) \in \mathscr{B}_0$. Let $\mathscr{D}(B) = \{\{x_k,n_k\}/\{n_k\} \to \infty,\ x_k \in H^{n_k}(B)\}$. Let $P(\{x_k,n_k\}) = \{x_k\}$. Let $\eta = \sup\{\beta(Ph/h \in \mathscr{D}(B)\}$. Note $\eta < \infty$ since $\gamma^+(B) \in \mathscr{B}_0$. We first show there is an $h^* = \{x_k^*,n_k^*\} \in \mathscr{D}(B)$ such that $\beta(Ph^*) = \eta$. Let $\{h_j\} \subset \mathscr{D}(B)$ be a sequence with $\beta(Ph_j) \to \eta$. Let $\hat{h}_j = \{(x_k,n_k) \in h_j/n_k > j\}$. Let $h^* = \bigcup_{j=1}^{\infty} h_j$ reordered in any way. Then we have $h^* \in \mathscr{D}(B)$ and so $\eta \geq \beta(h^*) \geq \beta(\hat{h}_j) = \beta(h_j) \to \eta$ as $j \to \infty$. Hence, $\beta(h^*) = \eta$.

Now for each $(x_k^*,n_k^*) \in h^*$ there is a set $\{x_k^{j*}, n_k^* - 1\}_{j=1}^{m_k} \subset H^{n_k^*-1}(B) \times Z$ such that $x_k^* \in H(\{x_k^{j*}\})$. Let

$g^* = \bigcup_{k=1}^{\infty} \{x_k^{j*}, n_k^* - 1\}_{j=1}^{m_k} \in \mathscr{D}(B)$. Hence, $\eta \geq \beta(g^*) \geq \beta(Hg^*) \geq \beta(h^*) = \eta$ with equality if and only if $\beta(g^*) = 0$. Hence, $\eta = 0$.

Now it is easy to see that there is a compact set $J \subset X_1$ such that $H^n(B) \to J$ in the Hausdorff metric. One may use Lemma 3.1 in Massatt [10].

<u>Proof of Theorem 4.3</u>: Let $||x||$ be the norm of x in X_1 or the distance from zero under some metric for X. Let $|x|$ be the sup norm for any $x \in C_B$. Let $B_N = \{x \in C_B / x(0) = 0,\ |x| < N\}$. If $A \subset X$, let $||A|| = \sup\{||x||: x \in A\}$. Pick $\sigma_n \to \infty$ such that $\sigma_1 = 0$ and $||B_n^{\sigma_n}|| < \frac{1}{2^n}$ for $n \geq 2$. Let $h(t) = (t+\sigma_{n+1})/n(\sigma_{n+1}-\sigma_n) - (\sigma_n+t)/(n+1)(\sigma_{n+1}-\sigma_n)$, i.e. $h(-\sigma_n) = \frac{1}{n}$ and $h(\cdot)$ connects these points with straight lines. Hence, we also have $h(\cdot)$ strictly increasing. This allows us to define two Banach spaces.

$$X_2 = \{x \in C(-\infty,0] / ||x||_2 = \sup_{\theta \in (-\infty,0)} |h(\cdot)x(\cdot)| < \infty,$$
$$\lim_{\theta \to -\infty} |h(\cdot)x(\cdot)| = 0\}.$$

$$X_1 = \{x \in W_1^{loc}(-\infty,0] / ||x||_1 = \sup_{\theta \in (-\infty,0)} |h^{1/2}(\cdot)x(\cdot)| +$$
$$||h^{1/2}(\cdot)\dot{x}(\cdot)||_\infty < \infty\}.$$

It is clear C_B is dense in X_2. Any Cauchy sequence of elements in C_B with $||\cdot||_2$ is Cauchy in X. Hence, we may consider $X_2 \to X$ with a continuous imbedding. From Arzela-Ascoli's theorem, we also get $i: X_1 \hookrightarrow X_2$ is a continuous, compact imbedding.

Under the general axioms imposed on X above, we cannot apply Theorem 2.1 directly, nor can we expect to get the solution map to be bounded dissipative in X_1. However, the proof will be in the same spirit.

As in Theorem 4.1, let $T_f(\omega): X \to X$ be the solution map from initial time 0 to time ω, $T_f(\omega)\phi = x_\omega(\cdot,\phi)$. Clearly, $T_f(\omega): X_i \to X_i$. We assume $T_f(\omega)$ maps bounded sets to bounded sets. The removal of this restriction involves only technical details. Let $C\phi = \phi^\omega - \phi(0)$ where $\phi(0)$ is the constant function. Let $U\phi = T_f(\omega)\phi - C\phi$. Since f is completely continuous, by arguments similar to the last theorem, we get $U: (X_1, \mathscr{F}_2) \to (X_1, \mathscr{F}_1)$ maps bounded sets to bounded sets. One problem we immediately encounter is that C may not be a contraction under any equivalent norm.

The proof that $U: X_1 \to X_1$ is completely continuous is identical to the last theorem. Our goal now will be to show that in X_1 we have sets S_0, S_1, and S_2 satisfying all the hypotheses of Theorem 4.2.

Let $A \subset X_1$ be any bounded set and let $A_r = \{x(\cdot): [-r,0] \to \mathbb{R}^n / x(\cdot)$ is the restriction on $[-r,0]$ of a function in $A\}$. Let $r(A) = \sup\{r / \mathrm{Cl}\, A_r$ is compact in $W_1^\infty[-r,0]\}$. Let $\beta(A) = \frac{1}{1+r(A)}$. Hence $\beta: \mathscr{B}_1 \to [0,\infty)$ is well-defined.

Let $X_0 = \{x \in W_1^{loc}(-\infty,0] / ||x||_0 = \sup|h(\cdot)^{1/3}x(\cdot)| + ||h(\cdot)^{1/3}\dot{x}(\cdot)||_\infty < \infty\}$. The same arguments as before show

$U: X_0 \to X_0$ and $U: (X_0, \mathcal{F}_2) \to (X_0, \mathcal{F}_0)$ maps bounded sets to bounded sets.

Let $B_R^{\mathbb{R}^n}$ dissipate points. Let $B_R^{C_B} \subset B^0_{Q(R)}$. If $x \in \text{Cl } B^0_{Q(R)}$ then x is clearly dissipated by $\text{Cl } B^0_{Q(R)}$. Let $y \in \text{Cl } B^0_{Q(R)}$ be chosen so that for all $t \in (-\infty,0]$ we have $h(t)^{1/3}y(t) = Q(r)$. Let $z(\cdot) = y(\cdot) - y^1(\cdot)$. Then $z(\cdot) > 0$ since $h(\cdot)$ is strictly monotone increasing. This defines a set $A = \{x(\cdot)/|x(t)| < z(t)$ for all $t \in (-\beta,0]\}$. Then any $x \in \text{Cl}(B^0_{Q(R)} + A)$ is also dissipated by $\text{Cl } B^0_{Q(R)}$. Now for every $x \in \text{Cl}(B^0_{Q(R)} + A)$ there exists an $n > 0$, $\varepsilon > 0$ such that $T^n_f(\omega)\{(\{x\} + B^2_\varepsilon) \cap \text{Cl}(B^0_{Q(R)} + A)\} \subset \text{Cl } B^0_{Q(R)}$. Because $\text{Cl}(B^0_{Q(R)} + A)$ is compact in X_2 there is a finite subcover of such sets and a maximum N such that n_i corresponding to the finite subcover satisfy $n_i \leq N$. Then, clearly $\gamma^+(\text{Cl } B^0_{Q(R)} + A) = \bigcup_{n=1}^{N} T^n_f(\omega)(\text{Cl } B^0_{Q(R)} + A)$.

Now, we may replace A by $A_N = \{x(\cdot)/|x(t)| < z(t)$ for all $t \in (-N,0]$ and $x(t) = 0$ for $t \in (-\infty,-N]\}$. Then we have

$$\gamma^+(\text{Cl } B^0_{Q(R)} + A_N) = \bigcap_{n=1}^{N} T^n(\omega)(\text{Cl } B^0_{Q(R)} + A_N).$$

Let $D = \text{Cl } B^0_{Q(R)} + A_N$. Define the set operator $H: \mathcal{B}_0 \to \mathcal{B}_0$ by $H(B) = \text{co}[T_f(\omega)\gamma^+(B \cap D)]$. The map H is β-condensing and type 2. H, X_0, X_1, and β satisfy all the properties of Theorem 4.4. Hence, H is asymptotically smooth. Because of the continuity, $\bar{H}: \mathcal{B}_0 \to \mathcal{B}_0$ defined by $\bar{H}(B) = \text{Cl co}[T_f(\omega)\gamma^+(B \cap D)]$ is also asymptotically smooth. Let $E = \bigcap_{n=1}^{\infty} \bar{H}^n(D)$. This is nonempty, bounded in X_0, compact in X_1, and convex. Now, let $S_0 = \mathrm{Cl}\, B^0_{Q(R)} \cap E$, $S_1 = D \cap E$, and $S_2 = E$. Applying Theorem 4.3 we get a fixed point of $T_f(\omega)$. This implies a periodic solution to (*) of period ω. This completes the proof.

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